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Non-parallel effects in the instability of Long's vortex

By M. R. FOSTER† AND DAVID JACQMIN‡
Ohio Aerospace Institute, Brook Park, OH 44142, USA

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As shown in Foster & Smith (1989), at large flow force M, Long's self-similar vortex is in the form of a swirling ring-jet, whose axial velocity profile is of sech^2 form. At azimuthal wavenumber n of comparable order to the axial wavenumber, linear helical modes of instability are essentially those of the Bickley jet varicose and sinuous modes. However, at small axial wavenumbers, the three-dimensionality of the vortex is important, and the instabilities depend heavily on the effects of the swirl. We explore here the effects of finite Reynolds number Re on these long-wave inertial modes. It is shown that, because the radial velocity scales with $Re^{-1}M$, the non-parallelism of the flow is more important than the viscous terms in determining the finite-Re behaviour. The three-layer structure of the parallel-flow instability modes remains, but with a critical layer considerably modified by radial velocity. In investigating the critical range $Re = O(M^3)$, we find the following: for $n > 1$, the non-parallelism stabilizes the unstable inertial modes, leading to determination of neutral curves; for $n < -1$, the non-parallel effects always destabilize the vortex to these helical modes. Determination of the unstable modes and neutral curves for the $n > 1$ case requires a computational scheme that accounts for the presence of viscosity. It turns out that the $n > 1$ ($n < -1$) modes are prograde (retrograde) with respect to the rotation of the main vortex.

1. Introduction

In the past twenty years, a number of new results have been obtained for the stability of three-dimensional vortices, that is, for concentrated vortices which have significant axial velocity shear. Building on the foundation of the works of Howard & Gupta (1962) and Drazin & Howard (1966), a series of papers by Lessen and colleagues (Lessen, Desphande & Hadji-Ohanes 1973; Lessen & Singh 1973; Lessen, Singh & Paillet 1974) appeared dealing for the most part with helical instabilities of the Batchelor vortex; Duck & Foster (1980) continued that effort. The particular vortex solution found by Long (1961), which has some geophysical significance (see Burggraf & Foster 1977), and has unstable helical modes not too different from the Batchelor vortex, has been studied by Foster & Duck (1982) and recently by Foster & Smith (1989).

The concern of this paper is to show that asymptotic results obtained by Foster & Smith for inviscid instability modes of the Type-II Long's vortex may be extended

† Permanent address: Department of Aeronautical and Astronautical Engineering, The Ohio State University, Columbus, OH 43210-1276, USA.

‡ Permanent address: NASA Lewis Research Center, Mail Stop 5-11, 21000 Brook Park Road, Cleveland, OH 44135, USA.

to account for effects of finite Reynolds number. Foster & Smith's asymptotic approach is applicable provided that both the vortex's Reynolds number and its flow force M (to be defined later) are large.

For large flow force, axial motion in the Type-II Long's vortex is confined to a thin ring-jet of large radius, $O(M)$, from the vortex axis (Foster & Smith 1989). Across this the swirl climbs rapidly from zero to a potential-vortex structure. A weak downflow occurs over the large central core. Instability modes are concentrated in the ring-jet region.

The inviscid modes for this vortex are centrifugal since they involve a radial-pressure-gradient/centrifugal-force imbalance; further, the absolute axial momentum and angular momentum of a base vortex plus linear disturbance are both fixed for fluid particles carried along with the disturbance. In addition, the growth rates are small, so that the eigenfunctions are close to the neutral mode whose eigenfunction is packed tightly about the critical layer, which occurs where

$$kW' + nG' = 0.$$

Here, the axial and azimuthal wavenumbers of the disturbance are k, n respectively; W, G are the axial velocity and angular momentum respectively in the vortex. Since G is monotone and W' may be either positive or negative, modes for both negative and positive values of n can occur. Hence, there are unstable eigenfunctions for $n > 0$ and $n < 0$ – though perhaps quite different. (Contrast, for example, the Batchelor vortex for which both W and G are monotone, and hence inertial modes occur only for $n < 0$.) Here, neutral disturbances undergo a bifurcation to unstable modes at a particular value of k. Since this critical layer occurs at large distance, $O(M)$, from the vortex axis, such instabilities are 'ring modes'. Such ring modes have been seen extensively in the recent works, particularly on the Batchelor vortex, utilizing high-wavenumber asymptotics. That work, begun by Leibovich & Stewartson (1983), and continued by them and others (Stewartson & Brown 1985; Stewartson & Capell 1985; Duck 1986; Stewartson & Leibovich 1987) showed similar concentration of the eigenfunction about a finite radial location, but for a different reason – because, there, n is large, and not M. The question of the inter-relationship of the large-n and large-M limits for Long's vortex is an interesting one, but not explored here. Long's vortex has the property that the scale of the radial velocity in the vortex is $O(Re^{-1})$ smaller than the axial velocity component. It is precisely $O(M Re^{-1})$ for large M, so that effects of finite viscosity come into the stability problem through the radial velocity before such an effect arises from friction terms explicitly. (The Reynolds number here is defined by $Re = \Gamma/\nu$, where Γ is the vortex circulation at its edge and ν is the fluid's kinematic viscosity.) This non-parallel flow exerts a damping influence for $Re = O(M^3)$, smoothing the bifurcation, and in the case of $n > 1$ modes makes it possible to determine the neutral curve for this vortex. For $n < -1$ modes, the effect of the radial velocity is to further destabilize the vortex; presumably the critical Reynolds number is $o(M^3)$ for $n < 0$.

Since for $Re = O(M^3)$ the modes computed in this paper are essentially inertial, the comments by Lin (1955) are relevant, and for every unstable, non-parallel mode computed, there is an adjoint mode (not necessarily a simple conjugate in this case). Computing growth-rate curves, then, requires special care for damped modes.

The plan of this paper is as follows. In §2, we review the basic vortex solution given by Long (1961), the special large-M case. In §3, we modify the formulation of the large-wavenumber inertial modes for this large-M vortex to include effects of finite, but large, Reynolds number. The two-point boundary-value problem that arises when including the dominant finite-Re term (non-parallelism) cannot be solved by hand. Therefore, in §4, we obtain the numerical solution. Discussion of the difficulty of continuing the ($n > 1$) solutions to damped cases is provided. In the Appendix, we show that sufficiently close to the point of bifurcation the viscous term is essential in obtaining the (unstable or stable) solutions. We expect that there is considerable generality to the analysis done here for a particular vortex. The instability modes studied propagate along and around the parent vortex in the same direction as the main rotation for $n > 1$ – 'prograde' modes – and opposite to the direction of the main rotation for $n < -1$ cases – 'retrograde' modes. So, from our analysis in this paper, the non-parallelism tends to stabilize (destabilize) the prograde (retrograde) modes. The large-M requirement of this work means simply that the vortex is dominated by its axial motion – the swirl is relatively less important. We are confident that very similar instability structures and their modification by non-parallelism will be in evidence for any parent vortex whose flux of axial momentum is large compared to its flux of angular momentum. In the setting of a geophysical vortex, for example, where the azimuthal wind speeds are relatively smaller than the vertical velocities, we expect that non-parallelism would make (unstable) retrograde modes much more likely to be observed.

2. The vortex flow

The vortex under study is due to Long (1961). Long found a similarity solution of the Navier–Stokes equations in which vortex velocities are given in terms of two functions $f(y)$ and $g(y)$:

$$U_d = \frac{\nu}{r_d}(yf'-f), \quad V_d = \frac{\Gamma}{2\pi r_d}g(y), \quad W_d = \frac{\Gamma^2}{8\pi^2 \nu z_d}\frac{f'(y)}{y}. \tag{2.1 a–c}$$

Here $\{U_d, V_d, W_d\}$ are the vortex radial, azimuthal, and axial velocities in a cylindrical polar coordinate $\{r_d, \theta, z_d\}$ system. Γ is the circulation of the vortex, and ν is the fluid viscosity. The similarity variable y is equal to $\Gamma r_d/2\sqrt{2\pi\nu z_d}$. The subscript d is used to denote dimensional variables.

The differential equations obeyed by f and g are, to $O(\nu/\Gamma)^2$,

$$yf'' - (1-f)f' - 4y^3 s = 0, \quad yg'' - (1-f)g' = 0, \quad 2y^3 s' + g^2 = 0, \tag{2.2 a–c}$$

where s is a pressure perturbation. Boundary conditions (see Long 1961) are

$$f(0) = f'(0) = g(0) = 0, \quad f'(\infty) = g(\infty) = 1. \tag{2.3 a, b}$$

Solutions can be characterized in terms of their 'flow force', whose non-dimensional form is

$$M = \pi \int_0^\infty [(f')^2 - g^2]\frac{dy}{y}. \tag{2.4}$$

Long and Burggraf & Foster (1977) found that solutions to (2.2)–(2.3) exist only for M larger than 3.75. There are then two solutions, denoted by Foster & Duck (1982) as Type I and Type II vortices. This paper considers instabilities of the Type II vortex at large values of M. Foster & Smith have shown that this vortex then has a relatively simple structure. At large M, the vortex is concentrated in a ring at $y = y_0 = 3\sqrt{2M/\pi}$. We introduce the small parameter $\epsilon = 1/y_0$. Away from the ring

$$f = \epsilon^{-1}F_0(Y) + O(\epsilon), \quad g = G_0(Y) + O(\epsilon^2), \quad s = \epsilon^2 S_0(Y) + O(\epsilon^4). \tag{2.5}$$

Here $Y = \epsilon y = y/y_0$. F_0, G_0, and S_0 are given by

$$F_0 = -Y^2/\sqrt{2}, \quad G_0 = 0, \quad S_0 = \tfrac{1}{4}, \quad Y < 1; \tag{2.6a}$$

$$F_0 = (Y^2 - \tfrac{1}{2})^{\frac{1}{2}}, \quad G_0 = 1, \quad S_0 = 1/(4Y^2), \quad Y > 1. \tag{2.6b}$$

Near $y = y_0$ we set $y = \epsilon^{-1} + \xi$. The inner asymptotic expansion for the ring vortex then takes the form

$$f = \epsilon^{-1}\mathscr{F}_0(\xi) + \mathscr{F}_1(\xi) + \ldots, \quad g = \mathscr{G}_0(\xi) + \epsilon\mathscr{G}_1(\xi) + \ldots, \quad s = \tfrac{1}{4}\epsilon^2 + \ldots. \tag{2.7}$$

As shown in Foster & Smith, the $(\;)_0$ terms are given by

$$\mathscr{F}_0 = \tanh(\hat{\xi})/\sqrt{2}, \quad \mathscr{G}_0 = \tfrac{1}{2}(1 + \tanh(\hat{\xi})], \tag{2.8}$$

where $\hat{\xi} = \frac{1}{2}(1 + \xi/\sqrt{2})$. The equation for $\mathscr{F}_1$ is

$$\mathscr{F}_1'' + (\mathscr{F}_0\mathscr{F}_1)' = -\xi\mathscr{F}_0'' + \mathscr{F}_0' + 1, \tag{2.9}$$

from which

$$\mathscr{F}_1 = \tfrac{1}{8}\xi^2 \operatorname{sech}^2(\hat{\xi}) + \sqrt{2}\xi \tanh(\hat{\xi}). \tag{2.10}$$

$\mathscr{F}_1$ is needed in order to match properly the radial velocity U_d. From (2.1*a*), U_d is given in the intense-jet region by

$$U_d = \frac{\nu}{r_d}\left[\frac{U_{00}}{\epsilon} + U_{01} + \ldots\right], \tag{2.11a}$$

$$U_{00} = \mathscr{F}_0'(\xi), \quad U_{01} = \xi\mathscr{F}_0' - \mathscr{F}_0 + \mathscr{F}_1'. \tag{2.11b}$$

Toward infinity $\mathscr{F}_0'$ vanishes strongly and the $O(1)$ terms $-\mathscr{F}_0 + \mathscr{F}_1'$ become dominant. It is these terms which match to the $O(1)$ terms of the outer expansion of U_d.

3. Instabilities

At moderate and short wavelengths, instabilities of the large-M Type-II Long's vortex are essentially the varicose and sinuous modes of the Bickley jet. Thus, these instabilities, both viscous and inviscid, are determined mostly by the vortex's sech² axial velocity. For long waves, however, the effect of the swirl velocity grows and the instability structures of both mode branches change significantly. In this paper, we deal with the viscous modifications to the inviscid long-wave modes discussed by Foster & Smith (1989).

The following analysis will deal with two distinct long-wave regimes. When the axial wavenumber k is $O(\epsilon)$ the three-dimensional character of the basic flow is important only in the regions outside the concentrated vortex ring. The critical-layer location, which is in the ring, is determined exclusively by the axial motion. As k drops to $O(\epsilon^2)$, three-dimensional effects extend into the vortex ring and the critical-layer location is then also affected by swirl.

The linearized instability equations are non-dimensionalized by setting $r_d = z_0\,\delta r$ and $z_d = z_0 + z_0\,\delta z$ and scaling time by $z_0^2\delta^3/\sqrt{2}\nu$, velocities by $\sqrt{2}\nu/z_0\delta^2$, and pressure by $2\nu^2/z_0^2\delta^4$. δ, an inverse Reynolds number, equals $2\sqrt{2}\pi\nu/\Gamma$. In terms of the scaled coordinates $y = r/(1 + \delta z)$.

A difficulty posed by the viscous Long's vortex is that it is non-parallel. To deal with that rationally, we suppose that the instability is of the form $A(r, \theta, z)\,e^{ikz}$ where

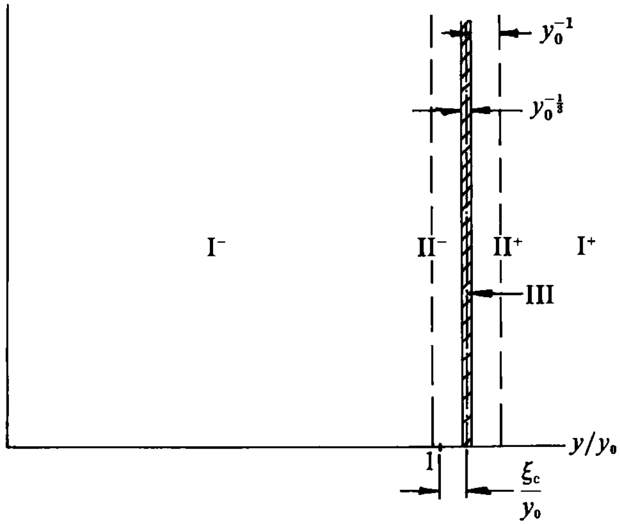

FIGURE 1. Regions of the asymptotic expansion: region II$^{\pm}$ is the ring-vortex region; III is the critical layer.

A is a slowly varying function of z. The region in z under consideration is of length $O(1/k)$. We constrain δ to be small enough so that (i) δz is $\ll 1$ and (ii) changes in the z-direction over the length $O(1/k)$ of A and of the basic-state flow are small. The second constraint turns out to be the significant one. We assume that the rate of change in the z-direction of the basic state and of A are of the same order. From the definition of y, the r- and z-derivatives of a basic-state quantity L are

$$\partial L/\partial r = (y/r)\,\mathrm{d}L/\mathrm{d}y \quad \text{and} \quad \partial L/\partial z = -(\delta y/(1+\delta z))\,\mathrm{d}L/\mathrm{d}y.$$

A Taylor series in z of L can thus be written as

$$L(r,z) = L(r,0) - \delta rz\,\partial L/\partial r|_{z=0} + O(\delta rz)^2. \tag{3.1}$$

The Long's vortex varies smoothly enough so that r-derivatives of its basic state are of the same order of magnitude as the basic state itself. The z-dependent terms in (3.1) can thus be neglected provided that $\delta rz \ll 1$. Since δz is small, $r \approx y$, and the region of interest in r, at the vortex ring, is at $r = O(\epsilon^{-1})$. δrz is therefore $O(\delta/\epsilon k)$. Setting $\delta = O(\epsilon^m)$, m must therefore be > 3 for $k = O(\epsilon^2)$ and > 2 for $k = O(\epsilon)$. In the subsequent analysis δ will be set to $O(\epsilon^4)$ for the first case and to $O(\epsilon^3)$ for the second.

With the above constraints, the scaled, linearized equations for the perturbation quantities $\{u(r,z), g(r,z), w(r,z), p(r,z)\}\,\mathrm{e}^{\mathrm{i}n\theta}\,\mathrm{e}^{\mathrm{i}k(z-ct)}$ about the location $z = 0$ are

$$\frac{1}{r}\frac{\partial ru}{\partial r} + \frac{\mathrm{i}n}{r}v + \mathrm{i}kw = R_p, \tag{3.2a}$$

$$\mathrm{i}\Phi u + U\frac{\partial u}{\partial r} + \frac{\partial U}{\partial r}(u - \delta rw) - 2\frac{V}{r}v = -\frac{\partial p}{\partial r} + \frac{\delta}{\sqrt{2}}\left(\nabla^2 u - \frac{2\mathrm{i}n}{r^2}v - \frac{1}{r^2}u\right) + R_u, \tag{3.2b}$$

$$\mathrm{i}\Phi v + U\frac{\partial v}{\partial r} + \frac{\partial V}{\partial r}(u - \delta rw) + \frac{U}{r}v + \frac{V}{r}u = -\frac{\mathrm{i}n}{r}p + \frac{\delta}{\sqrt{2}}\left(\nabla^2 v + \frac{2\mathrm{i}n}{r^2}u - \frac{1}{r^2}v\right) + R_v, \tag{3.2c}$$

$$\mathrm{i}\Phi w + U\frac{\partial w}{\partial r} + \frac{\partial W}{\partial r}(u - \delta rw) = -\mathrm{i}kp + \frac{\delta}{\sqrt{2}}\nabla^2 w + R_w, \tag{3.2d}$$

where $$\nabla^2 = \frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{n^2}{r^2}-k^2 \quad \text{and} \quad \Phi = kW+\frac{n}{r}V-kc.$$

The R in each equation is a remainder term that includes (i) Taylor series terms of basic-state flow quantities and (ii) z-derivatives of u, v, and w. For the chosen δ these terms are of order ϵ relative to the leading-order terms. The non-dimensionalized base-state velocities at $z = 0$ are

$$U = \frac{\delta}{\sqrt{2}r}\left(r\frac{\partial f}{\partial r}-f\right), \quad V = \frac{1}{r}g, \quad W = \frac{1}{\sqrt{2}r}\frac{\partial f}{\partial r}. \tag{3.3}$$

In the vortex ring W and its r-derivative are $O(1)$, V and its derivative are $O(\epsilon)$, and U and its derivative are $O(\delta/\epsilon)$.

In the following, we determine approximate long-wave solutions to (3.2) using the technique of matched asymptotic expansions. The solutions have three distinct regions (see figure 1); an outer potential flow region (region $\mathrm{I}^{\pm}$), the inner ring-vortex region (region $\mathrm{II}^{\pm}$), and a critical layer (region III).

3.1. *Instabilities for* $k = O(\epsilon^2)$

In this section, we consider the case $k = O(\epsilon^2) = \epsilon^2\beta$. δ is chosen to be $O(\epsilon^4)$. Setting $r = Y/\epsilon$ and utilizing the solutions (2.6) in region I where $Y = O(1)$, we obtain outer solutions valid to leading order in ϵ,

$$p^- = \epsilon q^- Y^{|n|}, \quad u^- = -\mathrm{i}q^-|n|Y^{|n|-1}/\beta c_0, \quad Y < 1; \tag{3.4}$$

$$p^+ = \epsilon q^+(n/Y^2-\beta c_0)\,Y^{-|n|}, \quad u^+ = -\mathrm{i}q^+|n|Y^{-|n|-1}, \quad Y > 1. \tag{3.5}$$

These solutions are derived in detail in Foster & Smith (1989). c_0 is the leading term in an asymptotic expansion in ϵ for the wave speed c. We restrict our discussion to instabilities for which $n \neq \pm 1$; the $|n| = 1$ solutions seem to have a different, specialized structure.

Quantities q^+ and q^- in (3.4)–(3.5) must be related by analysing the jet-like zone II and the critical-layer zone III. In region II, (3.4) and (3.5) are no longer the correct solutions to (3.2) because in this zone, the jet-like zone of the vortex, the velocity profiles are given by (2.7) instead of (2.6). Thus, we set $r = y_0+\xi$. The dependent variables are rescaled as $p = \epsilon\tilde{p}$, $u = \tilde{u}$, $v = \epsilon^{-1}\tilde{v}$ and $w = \epsilon^{-2}\tilde{w}$. The scalings for p and u are determined by the outer solution. Finally, we set $\Phi = \epsilon^2\tilde{\Phi}$. Substitution into (3.2) together with $k = \epsilon^2\beta$ yields the leading-order equations:

$$\frac{\mathrm{d}\tilde{u}_0}{\mathrm{d}\xi}+\mathrm{i}n\tilde{v}_0+\mathrm{i}\beta\tilde{w}_0 = 0, \quad \frac{\mathrm{d}\tilde{p}_0}{\mathrm{d}\xi}-2G_0\tilde{v} = 0, \tag{3.6a, b}$$

$$\mathrm{i}\tilde{\Phi}_0\tilde{v}_0+G_0'\tilde{u}_0 = 0, \quad \mathrm{i}\tilde{\Phi}_0\tilde{w}_0+W_0'\tilde{u}_0 = 0, \quad \tilde{\Phi}_0 = \beta(W_0-c_0)+nG_0. \tag{3.6c–e}$$

The primes indicate derivatives with respect to r. $W_0 = \operatorname{sech}^2(\hat{\xi})/4\sqrt{2}$ and $G_0 = \frac{1}{2}(1+\tanh(\hat{\xi}))$ are the $O(1)$-accurate approximations to W and to the base-state angular momentum $G = rV$ in the vortex ring.

The solution of (3.6) gives $\tilde{u}_0 = \text{constant}\times\tilde{\Phi}_0$, so, in a notation consistent with Foster & Smith (1989), we have

$$\tilde{u}_0^{\pm} = \frac{A_0^{\pm}}{\beta}\tilde{\Phi}_0, \quad \tilde{p}_0^{\pm} = \Pi^{\pm}+\mathrm{i}A_0^{\pm}\,\mathrm{i}G_0^2/\beta, \tag{3.7a, b}$$

where the $\pm$ notation refers to solutions in regions II^+ and II^-. The A_0 and Π

quantities are constants. Matching (3.7) to (3.4)–(3.5) leads to solutions in region II in terms of only q^- and q^+,

$$\tilde{u}_0^+ = \frac{-\mathrm{i}q^+|n|}{(n-\beta c_0)}\tilde{\Phi}_0, \quad \tilde{p}_0^+ = q^+(n-\beta c_0)\left[1+\frac{|n|(G_0^2-1)}{(n-\beta c_0)^2}\right], \tag{3.8a}$$

$$\tilde{u}_0^- = \frac{\mathrm{i}q^-|n|}{(\beta c_0)^2}\tilde{\Phi}_0, \quad \tilde{p}_0^- = q^-\left[1-\frac{|n|G_0^2}{(\beta c_0)^2}\right]. \tag{3.8b}$$

The region-II asymptotic expansion whose leading solutions are given by (3.8) fails at the critical layer, which, as discussed in Foster & Smith, occurs where both $\tilde{\Phi}_0$ and $\tilde{\Phi}_0'$ vanish. Let the location of the critical layer be denoted by ξ_c. The location ξ_c is the solution of the equation

$$\tanh(\xi_c/2^{\frac{3}{2}}+\tfrac{1}{2}) = \sqrt{2n/\beta}, \tag{3.9a}$$

and the wave speed, from (3.6*e*) is then

$$c_0 = 2^{-\frac{5}{2}}\left(1+\frac{\sqrt{2n}}{\beta}\right)^2. \tag{3.9b}$$

Hence, it is evident that the structure examined here requires that $\beta \geqslant \sqrt{2|n|}$, which we take to be so throughout the subsequent analysis.

Introducing $\rho = (\xi-\xi_c)/\epsilon^{\frac{1}{3}}$ and expanding c as $c_0+\epsilon^{\frac{2}{3}}c_1+\dots$, Φ becomes locally $\epsilon^{\frac{2}{3}}(\frac{1}{2}\tilde{\Phi}_c''\rho^2-\beta c_1)+O(\epsilon^3)$. Also we set $u=\epsilon^{\frac{2}{3}}u_c+\dots$, $p=\epsilon p_c+\dots$, $v=\epsilon^{-\frac{2}{3}}v_c+\dots$, and $w=\epsilon^{-\frac{5}{3}}w_c+\dots$. Further, set $U(\xi_c)=(\delta/\epsilon)\,U_c=(\delta/\epsilon)\,U_{00}(\xi_c)+\dots$. Substituting these into (3.2), we find from (3.2*b*) that p_c is a constant. Equations (3.2*c*) and (3.2*d*) can be combined into a single equation for u_c forced by p_c,

$$\frac{\delta}{\epsilon^4}U_c\frac{\mathrm{d}^2u_c}{\mathrm{d}\rho^2}+\mathrm{i}[\tfrac{1}{2}\tilde{\Phi}_c''\rho^2-\beta c_1]\frac{\mathrm{d}u_c}{\mathrm{d}\rho}-\mathrm{i}\tilde{\Phi}_c''\rho u_c = -n^2p_c. \tag{3.10}$$

Evaluation of $\tilde{\Phi}_c''$ and U_c then leads to the equation

$$\frac{\delta}{\epsilon^4}\frac{\sigma_-\sigma_+}{4}\frac{\mathrm{d}^2\phi}{\mathrm{d}\rho^2}-\mathrm{i}[\beta c_1+\Psi\rho^2]\frac{\mathrm{d}\phi}{\mathrm{d}\rho}+2\mathrm{i}\Psi\rho\phi = 0, \tag{3.11}$$

where
$$\sigma_\pm = 1\pm\sqrt{2n/\beta}, \quad \Psi = \frac{\beta}{32\sqrt{2}}(\sigma_-\sigma_+)^2, \tag{3.12}$$

and we have used the transformation

$$u_c = (\beta c_1+\Psi\rho^2)\frac{2n^2q^-[1-8|n|/(\beta\sigma_+)^2]}{\Psi\sigma_-\sigma_+(\delta/\epsilon^4)}+\left[\frac{2n^2[1-8|n|/(\beta\sigma_+)^2]}{\sigma_-\sigma_+(\delta/\epsilon^4)}-\frac{\mathrm{i}|n|}{\sqrt{2}\beta}\left(\frac{\sigma_-}{\sigma_+}\right)^2\right]q^-\phi. \tag{3.13}$$

Equation (3.11) is to be solved subject to the conditions

$$\phi \sim \rho^2, \quad \rho\to-\infty, \tag{3.14a}$$

$$\phi \sim A\rho^2, \quad \rho\to+\infty, \tag{3.14b}$$

where the quantity A is given by

$$A = \frac{1+\dfrac{\beta\sigma_-^3\sigma_+^3}{\beta^2\sigma_-^4-32|n|+8|n|\sigma_+^2}\Omega}{1+\dfrac{\beta\sigma_+\sigma_-^3}{8|n|-\beta^2\sigma_+^2}\Omega}, \quad \Omega = \frac{\mathrm{i}}{2\sqrt{2|n|}}\frac{\delta}{\epsilon^4}. \tag{3.14c}$$

u_c and ϕ must behave quadratically for large ρ in order to match to (3.8).

The foregoing analysis neglects the additional non-parallel remainder terms in (3.2). The presence of z in these terms complicates, but does not make impossible, a Fourier transform analysis. After transformation, (3.2) then becomes a set of partial differential equations in the independent variables r and β (or k). The dominant term added to (3.2) takes the form

$$\eta \frac{\delta}{\epsilon^2}\frac{\partial}{\partial \beta}\left[(nG_0''+\beta)\frac{\partial u}{\partial r}\right]. \tag{3.15}$$

Careful examination of this term and others in (3.2) shows that in the critical layer the ratio of (3.15) to the leading terms of (3.2) is $O(\delta/\epsilon^{\frac{8}{3}})$. However, the radial velocity term (the second term in (3.9)) is $O(\delta/\epsilon^4)$. Hence, we conclude that the remainder terms in (3.2) are negligible provided that $\beta \gg \epsilon^{\frac{4}{3}}$.

3.2. *Instabilities for $k = O(\epsilon)$*

For this case, we choose $\delta = O(\epsilon^3)$ and set $k = \epsilon\alpha$. The region-I solutions, as shown by Foster & Smith (1989), are then

$$p^- = I_{|n|}(\alpha Y), \quad u^- = \frac{1}{\mathrm{i}c_0} I'_{|n|}(\alpha Y), \tag{3.16a}$$

$$p^+ = qK_{|n|}(\alpha Y), \quad u^+ = \frac{q}{\mathrm{i}c_0} K'_{|n|}(\alpha Y). \tag{3.16b}$$

The quantity c_0 is, as before, the first term in an asymptotic series for c; q is an as-yet-unknown constant. $I_{|n|}$ and $K_{|n|}$ are modified Bessel functions of order $|n|$.

The analysis in region II proceeds much as in the former case, the chief difference being that the importance of the swirl is much less. We set $p = \tilde{p}$, $u = \tilde{u}$, $v = \tilde{v}$, and $w = \epsilon^{-1}\tilde{w}$. Φ is set to $\epsilon\tilde{\Phi}$, with $\tilde{\Phi}_0 = \alpha(W_0 - c_0)$. c is expanded as $c_0 + \epsilon^{\frac{2}{3}}c_1 + \dots$ and $\tilde{\Phi}$ as $\tilde{\Phi}_0 + \epsilon^{\frac{2}{3}}\tilde{\Phi}_1 + O(\epsilon)$. From (3.2*b*), $\tilde{p}_0$ is constant and from (3.2*a*, *d*) $\tilde{u}_0^{\pm} = (A_0^{\pm}/\alpha)\,\tilde{\Phi}_0$. Then $\tilde{\Phi}_1 = -\alpha c_1$. Substitution of $u = \tilde{u}_0 + \epsilon^{\frac{2}{3}}\tilde{u}_1 + \dots$ into (3.2) gives

$$\tilde{u}_1 = (A_0^{\pm}/\alpha)\,\tilde{\Phi}_1 = -A_0^{\pm}\,c_1. \tag{3.17}$$

As before, region III is located about $\tilde{\Phi}_0 = \tilde{\Phi}_0' = 0$. $\tilde{\Phi}_0' = 0$ is equivalent to $W_0' = 0$, which, from $W_0 = \frac{1}{4}\mathrm{sech}^2(\hat{\xi})\sqrt{2}$, occurs at $\hat{\xi} = \frac{1}{2}(1+\xi/\sqrt{2}) = 0$. Thus $\xi_c = -\sqrt{2}$. Setting $\xi = -\sqrt{2} + \epsilon^{\frac{1}{3}}\rho$, Φ in the critical-layer region is

$$\Phi = -\left(\frac{\alpha}{32\sqrt{2}}\rho^2 + \alpha c_1\right)\epsilon^{\frac{5}{3}} = \epsilon^{\frac{5}{3}}\Phi_c. \tag{3.18}$$

We set $u = \epsilon^{\frac{2}{3}}u_c$ while keeping the region II scalings for the other variables. Substitution into (3.2*c*, *d*) yields

$$\mu\frac{\mathrm{d}^2 u_c}{\mathrm{d}\rho^2} + \mathrm{i}\Phi_c\frac{\mathrm{d}u_c}{\mathrm{d}\rho} - \mathrm{i}\Phi_c' u_c = -(n^2+\alpha^2)\,p_c. \tag{3.19}$$

Here, $\mu = -\sqrt{2}U_c\,\delta/\epsilon^3 = \delta/(4\epsilon^3)$. The critical-layer pressure is, as before, a constant. A similar substitution to that of (3.13) then alters (3.19) and associated matching conditions into

$$\frac{\delta}{4\epsilon^3}\frac{\mathrm{d}^2\phi}{\mathrm{d}\rho^2} - \mathrm{i}\left[\alpha c_1 + \frac{\alpha\rho^2}{32\sqrt{2}}\right]\frac{\mathrm{d}\phi}{\mathrm{d}\rho} + \mathrm{i}\frac{\alpha\rho}{16\sqrt{2}}\phi = 0, \tag{3.20}$$

$$\phi \sim \rho^2, \quad \rho \rightarrow -\infty, \tag{3.20b}$$

$$\phi \sim A\rho^2, \quad \rho \rightarrow +\infty, \tag{3.20c}$$

where
$$\Lambda = \frac{n^2+\alpha^2-\surd 2\mu \mathrm{i} K'_{|n|}(\alpha)/(K_{|n|}(\alpha))}{n^2+\alpha^2-\surd 2\mu \mathrm{i} I'_{|n|}(\alpha)/(I_{|n|}(\alpha))}. \tag{3.20d}$$

For $\alpha \rightarrow 0$, asymptotic formulae for the Bessel functions give

$$\Lambda \sim 1+\frac{\mathrm{i}}{\surd 2|n|\beta}\frac{\delta}{\epsilon^4}. \tag{3.21}$$

This agrees with (3.14c) for β large. Note, further, that (3.20a) for $\alpha \rightarrow 0$ goes over to (3.11) for $\beta \rightarrow \infty$.

The coefficients of (3.20a) are fully contained in the coefficients of (3.11), so the same equation may be solved in both k regimes. Further, a composite formula for Λ may be constructed by putting together the formula (3.14c) with (3.20d). The resulting equation is

$$\Lambda = \frac{n^2+\alpha^2-\dfrac{\mathrm{i}}{2\surd 2\,\epsilon^3}\dfrac{\delta\, K'_{|n|}(\beta\epsilon)}{K_{|n|}(\beta\epsilon)}\dfrac{\sigma_-^3\,\sigma_+^3}{\sigma_-^4+|n|(\sigma_+^2-4)\,8/\beta^2}}{n^2+\alpha^2-\dfrac{\mathrm{i}}{2\surd 2\,\epsilon^3}\dfrac{\delta\, I'_{|n|}(\beta\epsilon)}{I_{|n|}(\beta\epsilon)}\dfrac{\sigma_+\sigma_-^3}{\sigma_+^2-8|n|/\beta^2}}. \tag{3.22}$$

4. Results

4.1. *General solution behaviour*

It is possible to rescale ρ so that (3.11) is transformed to an equation with only one parameter. Setting $\rho = [(\sigma_-\sigma_+/4\Psi)(\delta/\epsilon^4)]^{\frac{1}{3}}\hat{\rho}$, (3.11) becomes

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}\hat{\rho}^2}-\mathrm{i}(P+\hat{\rho}^2)\frac{\mathrm{d}\phi}{\mathrm{d}\hat{\rho}}+2\mathrm{i}\hat{\rho}\phi = 0, \tag{4.1}$$

where $P = \beta c_1(\delta\sigma_-\sigma_+/(4\epsilon^4))^{-\frac{2}{3}}\Psi^{-\frac{1}{3}}$. A similar equation would also replace (3.20). The fundamental solutions at large $\hat{\rho}$ are then, approximately, $s_1 = P+\hat{\rho}^2$ and $s_2 = \exp[\mathrm{i}(P\hat{\rho}+\frac{1}{3}\hat{\rho}^3)]/\hat{\rho}^4$. The second, depending on the sign of the imaginary part of P, is either exponentially decaying or growing as $\hat{\rho}$ goes to $+\infty$.

The exponential behaviour of the second solution has significant consequences. The boundary-value problem (3.11)–(3.14), (3.20) requires solutions that are dominated by the first fundamental solution at both plus and minus infinity. If $\mathrm{Im}\,(P) > 0$ the second fundamental solution decays at $+\infty$ and solutions can be found by starting at $-\infty$ with s_1 and integrating to $+\infty$. At $+\infty$ both s_1 and s_2 will be present but s_1 will automatically be dominant. Similarly, if $\mathrm{Im}\,(P) < 0$ the second fundamental solution grows at $+\infty$ and solutions can be found by starting at $+\infty$ with the first fundamental solution and integrating backwards. A major problem thus appears to arise in that solutions of (3.11)–(3.14), (3.20) are discontinuous from one sign of $\mathrm{Im}\,(P)$ to the other. Associated with this is the more minor puzzle of the non-uniqueness of the neutral modes. Then, since P is real and s_2 decays algebraically at both plus and minus infinity, both integration directions yield solutions.

These problems can be resolved by appealing to the fully viscous theory. We show in the Appendix that for relatively small values of δ/ϵ^4, the viscous term must be formally included to obtain solutions in the neighbourhood of $\mathrm{Im}\,(c_1) = 0$. In the context of the discussion of this section of the paper, the effect of the viscous forces is to damp the s_2 solution of (4.1) for $\rho \rightarrow +\infty$; the damping occurs independent of the sign of Im (P) and for any non-zero viscosity. A heuristic way to see this is to include viscous terms in the critical-layer analysis of §3. These terms add

$$-(\delta/\epsilon^{\frac{10}{3}})(1/\sqrt{2})(\mathrm{d}^3\phi/\mathrm{d}\rho^3)$$

to equation (3.11). Though of higher order than the other terms, it becomes important at large ρ as s_2 becomes increasingly oscillatory. Writing the exponential part of s_2 as $\exp[\kappa(\rho)\rho]$, κ shifts in phase from near $\frac{1}{2}\pi$ for ρ of $O(1)$ to $\frac{3}{4}\pi$ as $\rho\to\pm\infty$. In other words, s_2 moves from being oscillatory to being strongly damped.

The above argument resolves the proper direction of integration and thus the relevant solution type. With viscosity, s_2 grows rapidly as $\rho\to-\infty$. This eliminates from consideration solutions of (3.11) gained by integrating from $+\infty$. The above also points to a practical way to continue the solutions into parametric regions that have damped modes. Instead of integrating along the real line, we introduce a pseudo-viscosity by keeping slightly above it. This damps out the s_2 component of the solution and leaves the required s_1 component unchanged.

It turns out that the two solution types, the first found by integrating from $-\infty$ and the second found by integrating from $+\infty$, are related in a way that generalizes the conjugate mode pairs of inviscid parallel-flow theory. The first case gives the continuation of the unstable inviscid parallel-flow modes, while the second gives the continuation of the unphysical damped modes. Given a solution ϕ of (3.11)–(3.14), (3.20) then its conjugate ϕ^* satisfies

$$\frac{\mathrm{d}^2\phi^*}{\mathrm{d}\hat{\rho}^2}+\mathrm{i}(P^*+\hat{\rho}^2)\frac{\mathrm{d}\phi^*}{\mathrm{d}\hat{\rho}}-2\mathrm{i}\hat{\rho}\phi^*=0, \tag{4.2a}$$

$$\phi^*\sim\hat{\rho}^2,\quad \hat{\rho}\to-\infty, \tag{4.2b}$$

$$\phi^*\sim\Lambda^*\hat{\rho}^2,\quad \hat{\rho}\to+\infty. \tag{4.2c}$$

The substitution $\hat{\rho}\to-\hat{\rho}$ and the rescaling $\phi^*=\Lambda^*\bar{\phi}$ then gives

$$\frac{\mathrm{d}^2\bar{\phi}}{\mathrm{d}\hat{\rho}^2}-\mathrm{i}(P^*+\hat{\rho}^2)\frac{\mathrm{d}\bar{\phi}}{\mathrm{d}\hat{\rho}}+2\mathrm{i}\hat{\rho}\bar{\phi}=0, \tag{4.3a}$$

$$\bar{\phi}\sim\hat{\rho}^2,\quad \hat{\rho}\to-\infty, \tag{4.3b}$$

$$\bar{\phi}\sim\hat{\rho}^2/\Lambda^*,\quad \hat{\rho}\to+\infty; \tag{4.3c}$$

$\bar{\phi}$ therefore satisfies the original equation (4.1) except with P^* instead of P. Thus, solutions of the second type (the damped modes with Im $(P)<0$) can be found directly from those of the first type (the unstable modes with Im $(P)>0$) as their rescaled, flipped (about $\hat{\rho}=0$), conjugates.

4.2. *Numerical methods*

Numerical work was done using the scaled equation (4.1). The chief difficulty in solving it is the highly oscillatory nature of its second fundamental solution. Two methods have been used that overcome this problem. The first is the standard fourth-order Runge–Kutta technique but with adaptive gridding. Integration step sizes were made $O(1/\hat{\rho}^2)$ so as to keep numerical stability. The second is the Crank–Nicholson method, which is implicit and always stable, using fixed steps. Both were found to be adequate and gave little trouble. Results presented herein have been checked by calculating them using both methods. Another difficulty was starting the numerical integrations off. It was found that simply setting $\phi=\hat{\rho}^2$ at a large but still economical value of $-\hat{\rho}$ was insufficiently accurate. This resulted in the presence of a significantly large multiple of the unwanted second fundamental solution, as was observed in the numerical solution's oscillatory behaviour. Fortunately, a more accurate beginning solution, valid for large $-\hat{\rho}$, can be derived asymptotically from (3.11).

To $O(\hat{\hat{\rho}}^{-4})$, it is

$$s_1 = (1+\hat{\hat{\rho}}^2)\left(1 - \mathrm{i}\gamma\left(f(\hat{\hat{\rho}}) + \frac{\hat{\hat{\rho}}}{1+\hat{\hat{\rho}}^2}\right) - \gamma^2\left(\frac{\hat{\hat{\rho}}}{1+\hat{\hat{\rho}}^2} f(\hat{\hat{\rho}}) + \frac{1}{2}\frac{1}{1+\hat{\hat{\rho}}^2} - \frac{1}{2}\frac{1}{(1+\hat{\hat{\rho}}^2)^2} + \tfrac{1}{2} f(\hat{\hat{\rho}})^2\right)\right), \tag{4.4}$$

where $\gamma = \delta\sigma_- \sigma_+ \Psi^{\frac{1}{2}}/4\epsilon^4(\beta c_1)^{\frac{3}{2}}$, $\hat{\hat{\rho}} = \gamma^{\frac{1}{3}}\hat{\rho}$, and $f(\hat{\hat{\rho}}) = \frac{1}{2}\pi + \tan^{-1}\hat{\hat{\rho}}$. The result in (4.4) and its derivative were used to start off the numerical solution. With this, it was found easy to keep the unwanted oscillatory second fundamental solution down to a very low level, about 10^{-8} that of the wanted startup level.

No particular difficulty was experienced in numerically determining Λ. Integrations were simply continued until the oscillations of the $+\infty$ fundamental solution died out to a preset small level. Λ was then approximated by dividing the numerical solution by $P+\rho^2$ and averaging the result over the last few tens of integration steps.

4.3. *Neutral modes and some general results for stability and instability*

One way of solving (4.1) is by setting the physical parameters $(n, \beta, \delta, \epsilon)$, of the problem, thus determining Λ, and then solving (4.1) for various P (and thus c_1) until the P is found that yields $\phi(+\infty) = \Lambda\hat{\rho}^2$. An alternative approach is to integrate (4.1) for a set of discrete P, thus finding $\Lambda(P)$ over some range of P. $\Lambda(P)$, assuming care in accounting for possible multivaluedness, can then be inverted to find $P(\Lambda)$. From that, with further algebra, $c_1(n, \beta, \delta, \epsilon)$ can be determined. The second approach offers the possibility of needing many fewer integrations of (4.1) than the first.

The second approach is definitely the most efficient when searching for neutral modes. P is then on the real line and Λ generated by P is limited to curves in the complex plane. These curves are shown in figure 2 for $-8 \leqslant P \leqslant 9$. (At these points, the $|P| = \infty$ limits have essentially been attained.) For $P \to +\infty$, the $P(\Lambda)$-curve smoothly approaches $(1,0)$ on a tangent to the line $\Lambda = 1$. For $P \to -\infty$, $P(\Lambda)$ is on an infinite set of diminishing 'circles' that, again, reduce to $+1$ in their limit. $P(\Lambda)$ is multivalued in this region, for $\Lambda_r \geqslant 1$. Everywhere else, computations indicate that $P(\Lambda)$ is single-valued, with Λ inside the neutral curve indicating instability.

It turns out that only a small portion of these curves, in fact only (*a*) the section in $\Lambda_r \leqslant 1$, $\Lambda_i \geqslant 0$ and (*b*) the single point $\Lambda = (1,0)$, is actually relevant to this stability problem. Also, the multivalued region is unimportant. This is because of restrictions on Λ imposed by its formulae (3.14*c*) and (3.22). These restrictions also lead to some reasonably general results about parameter ranges of instability and stability.

We now look at these restrictions. Equation (3.14*c*) can be written as

$$\Lambda = \frac{1+t\Omega_1 \mathrm{i}}{1+b\Omega_1 \mathrm{i}}, \tag{4.5}$$

where $t = \beta\sigma_-^3\sigma_+^3/(\beta^2\sigma_-^4 - 32|n| + 8|n|\sigma_+^2)$ and $b = \beta\sigma_-^3\sigma_+^3/(8|n|\sigma_+^2 - \beta^2\sigma_+^4)$. t and b are functions of n and β, while Ω_1 is a function of only δ/ϵ^4. Ω_1 can vary from 0 to $+\infty$.

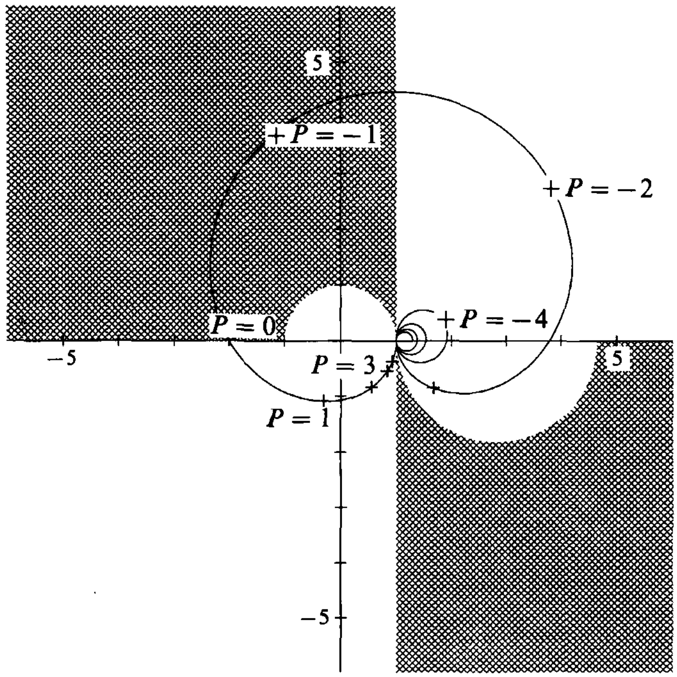

FIGURE 2. Neutral Λ-curves (Λ associated with neutral modes). $\Lambda(P_r)$ is shown for $-8 \leqslant P_r \leqslant 9$. The cross-hatching shows the Λ permitted by (3.14c) for $n > 1$.

Meanwhile, for a fixed n and β, the path of Λ describes a semicircle with diameter $|1-t/b|$ in the complex plane,

$$(\Lambda_r - \tfrac{1}{2}(1+t/b))^2 + \Lambda_i^2 = \tfrac{1}{4}(1-t/b)^2, \tag{4.6}$$

where Λ_r varies from 1 to t/b and $\operatorname{sgn}(\Lambda_i) = \operatorname{sgn}(t-b)$. It turns out to be fairly easy to establish bounds on the ratio b/t. These bounds then yield general bounds on the location and radii of the Λ semicircles, and thus on Λ.

We first consider $n < -1$. For this case it can be shown that $|b|$ is greater than or equal to $|t|$ and thus $|\Lambda| \leqslant 1$. Solutions intersect the neutral curves only in the inviscid limit as P goes to $+\infty$. As a result, *all* $\delta > 0$, $n < -1$ *solutions are unstable.*

For $n > 1$, it can be shown that $|t| > |b|$ and that therefore $|\Lambda| \geqslant 1$. There are two subcases. The two are separated by the value of β, $\beta = \beta^*$, for which t is singular (its denominator is zero) and Λ becomes infinite. For $\beta < \beta^*$, t is $\leqslant 0$ and for $\beta > \beta^*$, t is $\geqslant 0$; b is always $\leqslant 0$.

For $\beta < \beta^*$, $\Lambda_r \geqslant 1$ while Λ_i is bounded by

$$-\infty \leqslant \Lambda_i \leqslant -[(\Lambda_r - 1)((t/b)_{\min} - \Lambda_r)]^{\frac{1}{2}}. \tag{4.7}$$

The range of Λ for this case is shown in figure 2 by the cross-hatching in the fourth quadrant. For $n = 2$, $(t/b)_{\min} = 4.64$. For $n = 3$ the minimum t/b is 7.93, for $n = 4$ the minimum is 11.35. It can be seen that the range of Λ lies outside the neutral curves. Thus, *all* $\delta > 0$, $n > 1$, $\beta < \beta^*$ *modes are damped.*

The range of Λ for $\beta > \beta^*$ is shown in figure 2 by the cross-hatching in the second quadrant. The result can be stated as $\Lambda_r \leqslant 1$ with $|\Lambda| \geqslant 1$. This region includes part of the $\Lambda_r < 1$ neutral curve. *Only* $n > 1$, $\beta > \beta^*$ *gives neutral modes.* Such neutral modes are found starting at $\Lambda \approx (1, 4.474)$, $P \approx -1.421$, for $\beta = \beta^*$, and progressing to $\Lambda = (-2, 0)$, $P = 0$, with β such that $t/b = -2$. A rather surprising result is that the range of neutral Λ, P is independent of $n > 1$.

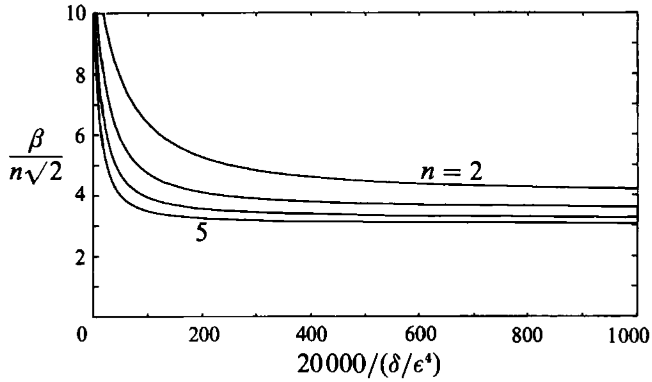

FIGURE 3. Neutral curves for $n = 2$, 3, 4, and 5. The curves are calculated using (3.14*c*).

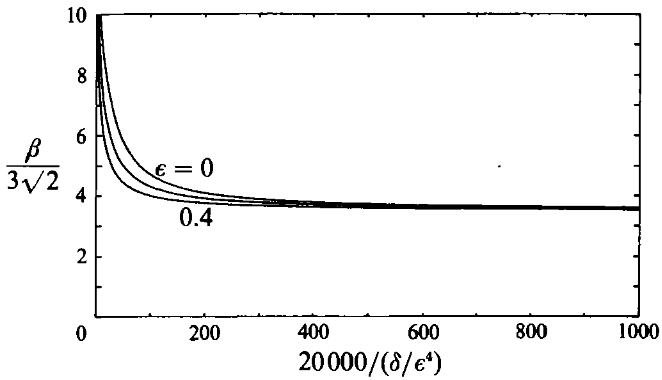

FIGURE 4. Neutral curves calculated from (3.22) for $n = 3$, for $\epsilon \rightarrow 0$ and $\epsilon = 0.2$, and 0.4. The $\epsilon \rightarrow 0$ case is equivalent to $\alpha \rightarrow 0$ and recovers (3.14*c*).

None of the above results are changed in a significant way for the more complete Λ equation (3.22). Most important, the range of relevant neutral Λ, and the P associated with that Λ range, remains the same. Equation (3.22) can be written

$$\Lambda = \frac{1+F_K t\Omega_\alpha \mathrm{i}}{1+F_I b\Omega_\alpha \mathrm{i}}, \tag{4.8}$$

where $$\Omega_\alpha = \frac{|n|}{2\sqrt{2(n^2+\alpha^2)}}\frac{\delta}{\epsilon^4}, \quad F_K = \left|\frac{\alpha K'_{|n|}(\alpha)}{|n|K_{|n|}(\alpha)}\right|, \quad F_I = \frac{\alpha I'_{|n|}(\alpha)}{|n|I_{|n|}(\alpha)}.$$

F_K and F_I obey the inequalities $F_K, F_I > 0$ and $F_K > F_I$. The second inequality leads to the result that for $n > 1$ the bounds on Λ are the same for (3.22) as for (3.14*c*). Bounds on Λ for $n < -1$ are changed in only minor ways. In particular, this case still has only unstable modes.

Figures 3 and 4 show positive-n neutral curves in the (Re, β) plane. (For fixed ϵ, the Reynolds number is proportional to ϵ^4/δ.) Figure 3 shows neutral curves for $n = 2, \ldots, 5$ for $\delta = O(\epsilon^4)$. This uses (3.14*c*) calculating Λ. Figure 4 shows results using (3.22). Results are shown for various ϵ for $n = 3$. It was originally hoped that the $k = \epsilon\alpha$ approximation would yield an upper branch for the neutral curve. However, as the figure shows and the previous analysis in this section indicates, even very large α and δ affect the neutral curves only slightly.

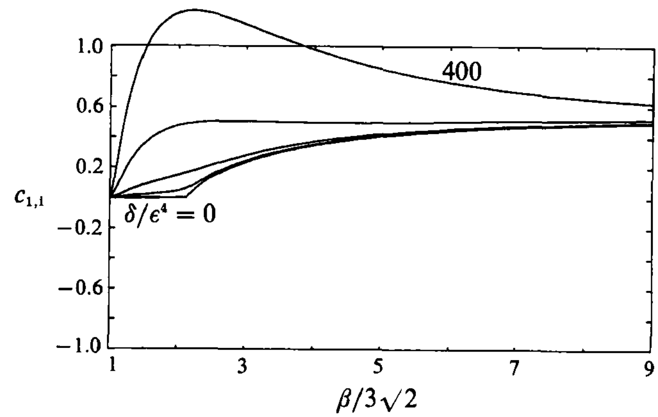

FIGURE 5. $c_{1,\mathrm{i}}$ for $n = -3$, for $\delta/\epsilon^4 = 0$, 5, 20, 100, and 400. β^* is located at the break in the $\delta/\epsilon^4 = 0$ curve. The β-axis is scaled by the minimum allowable value of β, which is $3\surd 2$ for $n = \pm 3$.

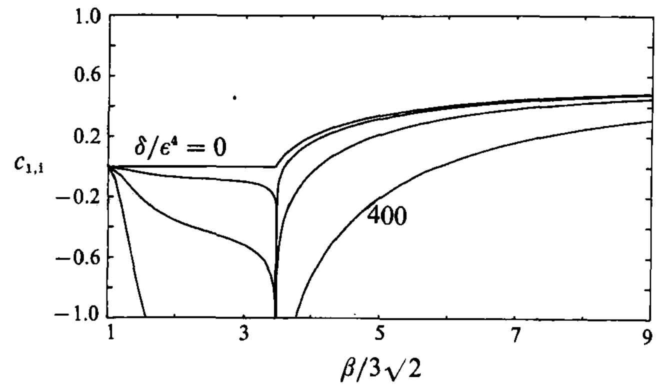

FIGURE 6. $c_{1,\mathrm{i}}$ for $n = 3$, for $\delta/\epsilon^4 = 0$, 20, 100, and 400. β^* is located at the break in the $\delta/\epsilon^4 = 0$ curve. The β-axis is scaled by the minimum allowable value of β, which is $3\surd 2$ for $n = \pm 3$.

4.4. *Growth rates*

Figures 5 and 6 show $c_{1,\mathrm{i}}$ for, respectively, $n = -3$ and 3, for various values of the viscous parameter δ/ϵ^4. The symbol $c_{1,\mathrm{i}}$ has been written for $\mathrm{Im}\,(c_1)$. Λ is calculated using (3.14*c*). The plots show that $c_{1,\mathrm{i}}$ is a monotone function of the viscous parameter. In agreement with the analysis of the previous section, increasing δ/ϵ^4 is destabilizing for $n < -1$ and stabilizing for $n > 1$. The singularity in $c_{1,\mathrm{i}}$ indicated in figure 6 is at $\beta = \beta^*$. As mentioned in the previous section this value of β makes Λ infinite. It is also the value at which the inviscid parallel-flow solution bifurcates from neutrality to instability. For $n < -1$, $\beta = \beta^*$ makes the denominator of (3.14*c*) singular, and thus makes $\Lambda = 0$. The asymptotics of both cases are discussed in detail in the next subsection.

Another significant result visible in the plots is the relative unimportance of non-parallel effects as β becomes very large. All the viscous curves then asymptote toward the inviscid parallel-flow curve. This result can be shown to hold by considering (3.11) and (3.14*c*). As $\beta \to \infty$, $\sigma_\pm \to 1$ and, from inspection of (3.14*c*),

$$\Lambda \sim 1 + 2\Omega/\beta. \tag{4.9}$$

Thus Λ approaches the inviscid parallel-flow value of 1 that is given by $\Omega \equiv 0$.

Meanwhile, (3.11) also simplifies to the inviscid case. Its zeroth- and first-derivative terms are multiplied by $\beta \to \infty$ while its second derivative is multiplied by δ/ϵ^4, which is taken here to be fixed. The solution to (3.11) for large β is given by (4.4) to $O(\beta^{-3})$ (remembering that $\gamma \propto \beta^{-\frac{3}{2}}$). For small γ, (4.4) is valid along the entire integration path. This solution is the inviscid parallel-flow solution plus a small ($O(\beta^{-\frac{3}{2}})$) non-parallel modification.

4.5. *The solution for* $\delta/\epsilon^4 \ll 1$

The parallel-flow growth rate is given in Foster & Smith (1989), and corresponds to the solution of (3.11)–(3.14) for $\delta/\epsilon^4 \equiv 0$. As it may be noted from (3.14*c*) above, the limit solution fails near singularities of Λ, denoted by β^*, previously discussed in §4.3. Those locations are the bifurcation points for the parallel-flow solution. In their neighbourhood, the growth rates depart significantly from the parallel-flow growth rates for arbitrarily-small δ/ϵ^4, as we shall see below. For $\delta/\epsilon^4 \equiv 0$, the solution of (3.11)–(3.14) for c_1 is†

$$(\beta c_1)^{\frac{3}{2}} = \frac{\pi n^2[(\beta^2/8|n|)\,\sigma_+^2 - 1]\,[(\beta^2/8|n|)\,\sigma_-^3 - \sigma_+ - 2]\,\sigma_+}{(32\beta/\sqrt{2})^{\frac{1}{2}}[1-\frac{1}{2}|n|(1+\frac{1}{4}[(\beta/\sqrt{2n})+(\sqrt{2n}/\beta)]^2)]}. \tag{4.10}$$

Figure 5, for example, shows the parallel-flow result ($\delta/\epsilon^4 \equiv 0$) as well as the growth rate for, say, $\delta/\epsilon^4 = 5$. Note that those two curves lie very close together except in the region of bifurcation. Thus, the solution to (3.11)–(3.14) goes uniformly to the inviscid solution except near the neutral condition, $\beta = \beta^*$; $\beta^* = 9.1416$ for $n = -3$.

Now, careful inspection of (3.14*c*) shows that $\Lambda \to 1 + O(\Omega)$ for $\Omega \to 0$ except in two situations, where the coefficients of Ω in the quotient for Λ are singular.

First, consider the case when the coefficient of Ω in the denominator of (3.14*c*) vanishes, namely,

$$\beta^2\sigma_+^2 - 8|n| = 0. \tag{4.11}$$

Note, from (4.10), that a β value that corresponds to a solution of (4.11) is associated with a neutral mode for the inviscid problem. Solution of (4.11) gives

$$\beta^* = (-2n)^{\frac{1}{2}}[2+(-n)^{\frac{1}{2}}], \quad n < 0 \tag{4.12}$$

(equation (4.11) has solutions only for $n < 0$). Near $\beta = \beta^*$, σ_- has the form

$$\sigma_-^* = 2\frac{1+(-n)^{\frac{1}{2}}}{2+(-n)^{\frac{1}{2}}}, \tag{4.13}$$

and Λ becomes

$$\Lambda^* = \left[1 + \frac{\sqrt{2}}{n}\frac{(1+(-n)^{\frac{1}{2}})^3}{(2+(-n)^{\frac{1}{2}})^3}\frac{\mathrm{i}\delta/\epsilon^4}{(\beta^*-\beta)}\right]^{-1}. \tag{4.14}$$

In the same region

$$\Psi \to \Psi^* \equiv \tfrac{1}{2}(-n)^{\frac{1}{2}}(1+(-n)^{\frac{1}{2}})^2/(2+(-n)^{\frac{1}{2}})^3. \tag{4.15}$$

Setting

$$\beta = \beta^* + (\delta/\epsilon^4)\,\hat{\beta}, \tag{4.16a}$$

$$\rho = \left(\frac{2\delta}{\epsilon^4}\frac{[2+(-n)^{\frac{1}{2}}]}{(-n)^{\frac{1}{2}}[1+(-n)^{\frac{1}{2}}]}\right)^{\frac{1}{3}}\rho^*, \tag{4.16b}$$

$$c_1 = 2^{-\frac{5}{3}}\frac{[1+(-n)^{\frac{1}{2}}]^{\frac{4}{3}}}{(-n)^{\frac{1}{3}}[2+(-n)^{\frac{1}{2}}]^{\frac{10}{3}}}\left(\frac{\delta}{\epsilon^4}\right)^{\frac{2}{3}}c_1^*, \tag{4.16c}$$

† This replaces equation (5.32) of Foster & Smith (1989), in which there is an algebraic error.

(3.11)–(3.14) are rescaled to

$$\frac{d^2\phi^*}{d\rho^{*2}} - i[c_1^* + \rho^{*2}]\frac{d\phi^*}{d\rho^*} + 2i\rho^*\phi^* = 0, \tag{4.17}$$

$$\phi^* \sim \rho^{*2}, \quad \rho^* \to -\infty, \tag{4.18}$$

$$\phi^* \sim \left[1 - \frac{\sqrt{2}}{n}\frac{(1+(-n)^{\frac{1}{2}})^3\, i}{(2+(-n)^{\frac{1}{2}})^3 \hat{\beta}}\right]^{-1} \rho^{*2}, \quad \rho^* \to +\infty. \tag{4.19}$$

Therefore, near β^*, the complete viscous problem must be solved even if the Reynolds number $1/\delta$ is large. Further, we see that within a region of width $O(\delta/\epsilon^4)$ near β^*, the imaginary part of c_1 is $O(\delta/\epsilon^4)^{\frac{2}{3}}$.

A very similar structure occurs also for $n > 0$ modes near where $\mathrm{Im}(c_1) \to 0$; it occurs where the Ω coefficient of the numerator of (3.14*c*) is singular, so

$$\beta^2\sigma_-^4 - 32|n| + 8|n|\sigma_+^2 = 0. \tag{4.20}$$

Writing Z for $\sqrt{2n}/\beta^*$ and substituting into (4.20) leads to the cubic equation

$$(n+4)Z^3 + 3(4-n)Z^2 + 3nZ - n = 0, \quad n > 0. \tag{4.21}$$

For $n > 1$, this cubic has one real positive root, hereafter denoted by $Z_*(n)$. For $n = 3$, for example, $Z_* = 0.2874$ and $\beta^* = 14.76$. As with the $n < 0$ case, we set $\beta = \beta^* + (\delta/\epsilon^4)\hat{\beta}$. Also, we put $\rho = [8(\delta/\epsilon^4)Z_*/(1-Z_*^2)]^{\frac{1}{3}}\rho^*$, and

$$c_1 = (\delta/\epsilon^4)^{\frac{2}{3}}[(Z_*(1-Z_*^2)^2)^{\frac{2}{3}}/8\sqrt{2}]\, c_1^*. \tag{4.22}$$

The equations for ϕ^* then have the same form as for the $n < 0$ case. In fact, (4.17), (4.18) carry over unchanged and (4.19) is replaced by

$$\phi^* \sim \left[1 + \frac{i}{16\sqrt{2n}\hat{\beta}}\frac{(1-Z_*^2)^2(1+Z_*)}{[n(Z_*-1)^2 - 4Z_*^2]}\right]\rho^{*2}, \quad \rho^* \to +\infty. \tag{4.23}$$

Thus, we verify from the structure of the boundary-value problem for ϕ that the non-parallel flow solution deviates substantially from the parallel-flow result near the bifurcation point, β^*, even for $\delta/\epsilon^4 \ll 1$! We see from figures 5 and 6, however, that the departure from the $\delta/\epsilon^4 \equiv 0$ growth rates is rather different in the two cases $n > 1$ and $n < -1$.

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Appendix

In §4.5, we examined the solutions of (3.11)–(3.14) in detail near $\beta = \beta^*$ for $\lambda \ll 1$, where we now define λ as δ/ϵ^4. In summary, we found for the $n > 0$ case that when β is within $O(\lambda)$ of β^*, region III shrinks by a factor $\lambda^{\frac{1}{3}}$, the growth rate $\mathrm{Im}(\beta c_1)$ is $O(\lambda^{\frac{2}{3}})$ and non-parallelism makes the solution significantly different from the $\lambda \equiv 0$ case given by (4.10). We show below that, in fact, for λ sufficiently small the viscous term dominates the non-parallel term; in that range, then, no proper $\lambda \ll 1$ solution, stable or unstable, may be obtained without the viscous term. We have not yet been able to show, with a similar argument, the necessity of the viscous term for obtaining the solution for λ not small.

We notice, from (4.10) and $n > 0$, that for $\beta \to \beta^*$, $c_1 \to 0$, in fact

$$\beta c_1 \sim K(Z_*)(\beta - \beta^*)^{\frac{2}{3}} e^{2\pi i/3}, \tag{A 1}$$

where the exact form of K does not matter at this point.

In the notation of (3.10), the exact parallel-flow solution to that equation (setting $U_c = 0$) is easily shown to be

$$u_c = C_1(\rho^2+a) - \frac{in^2 p_c}{\Psi}\left[\frac{(\rho^2+a)}{2a^{\frac{3}{2}}}\tan^{-1}\left(\frac{\rho}{a^{\frac{1}{2}}}\right) - \frac{\rho}{a^{\frac{1}{2}}}\right], \tag{A 2}$$

where a is written for convenience for $\beta c_1/\Psi$. Notice that as a (or $\beta c_1) \rightarrow 0$, the solution is singular, and also it seems reasonable to rescale ρ as well, so we write now $\rho = \Delta\bar{\rho}$ and $a = A\Delta^2$, where it is understood that A is complex-valued. Then, this substitution into (A 2) shows that the dominant behaviour of u_c for $\Delta \rightarrow 0$ is

$$u_c \sim -\frac{in^2 p_c}{2\Psi}\frac{A+\bar{\rho}^2}{A^{\frac{3}{2}}\Delta}\tan^{-1}\left(\frac{\bar{\rho}}{A^{\frac{1}{2}}}\right). \tag{A 3}$$

With these preliminaries, we now substitute the change in variable ρ into the partial differential equation (3.10), including the third-derivative (viscous) term that is small for the analysis in §3 of the unstable modes. In that case, (3.11) now becomes

$$L(u_c) = \frac{\sigma_-\sigma_+}{4\Psi^{\frac{1}{4}}}\left(\frac{\lambda}{\epsilon^2}\right)^{\frac{1}{4}}\frac{d^2u_c}{d\bar{\rho}^2} + i(\bar{\rho}^2+A)\frac{du_c}{d\bar{\rho}} - 2i\bar{\rho}u_c + \frac{d^3u_c}{d\bar{\rho}^3} = \frac{n^2p_c}{\Psi^{\frac{3}{4}}\lambda^{\frac{1}{4}}\epsilon^{\frac{1}{8}}}, \tag{A 4}$$

where the distinguished limit has been chosen as

$$\Delta = (\lambda\epsilon^{\frac{2}{3}}/\Psi)^{\frac{1}{4}}. \tag{A 5}$$

Equation (A 4) may be rewritten using a change of variable like (3.13) into the equation $L(\phi) = 0$, with matching conditions like (3.14*a*, *b*), viz.,

$$\phi \sim \bar{\rho}^2, \quad \bar{\rho} \rightarrow -\infty \tag{A 6a}$$

$$\phi \sim \left(1+\frac{d_0}{\beta}\left(\frac{\lambda}{\epsilon^2}\right)^{\frac{1}{4}}\right)\bar{\rho}^2, \quad \bar{\rho} \rightarrow \infty, \tag{A 6b}$$

where d_0 is a constant depending on Z_* and β has been written as $\beta^* + \lambda^{\frac{3}{4}}\epsilon^{\frac{1}{2}}\tilde{\beta}$. So, for all $\lambda = O(\epsilon^2)$ the viscous term is essential, and is needed to generate the proper solutions for $\mathrm{Im}(A) < 0$.

We summarize as follows. We noted above, for small λ, that for $\beta - \beta^* = O(\lambda)$, non-parallel corrections to (4.10) are important; here, we found that for $\beta - \beta^* = O(\lambda^{\frac{3}{4}}\epsilon^{\frac{1}{2}})$, the viscous term is dominant, and not non-parallelism. Since the first region of β non-uniformity occurs over a wider zone than the second for $\lambda \ll \epsilon^2$, we conclude that for $\lambda \ll \epsilon^2$, (A 4) is the appropriate equation to solve for u_c, and viscous effects are critical to determination of stable or unstable eigenmodes. For λ/ϵ^2 of order unity, (A 4) indicates that both viscous and non-parallel terms are required. For $\lambda \gg \epsilon^2$, non-parallelism becomes more important and the arguments of §4.5 are recovered. As noted in §4.1, it appears that the viscous term is essential for calculation of the proper inertial eigenmode for $\lambda = O(1)$, though we have not been able to extend the argument given here, which is restricted to relatively small values of λ.

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